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On the concepts of Lie and covariant derivatives of spinors: part II

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Dedicated to our colleague Professor P Barry on the occasion of his 60th birthday

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Abstract. The framework developed in part I for defining Lie and covariant differentiation of spinors is investigated to show its compatibility with tensor calculus. The special role played by conformal Killing vectors and conformal connections is also determined.

1. Introduction

In a previous article [1], hereafter referred to as 'part I', we presented a general formalism which enables one to define the concept of the covariant derivative $\nabla_X\psi$ of a spinor field ψ without any restriction on the spacetime connection. (As it was emphasized, these results can straightforwardly be re-interpreted as providing a definition of the Lie derivative $\mathcal{L}_X\psi$ of a spinor field ψ without restriction on the vector field X.) The literature, however, contains the claim [2] that 'we could only hope to define $\mathcal{L}_X\alpha^A$ for vector fields X^a satisfying $\mathcal{L}_X g_{ab} = kg_{ab}$. These are *conformal Killing vectors* and correspond to conformal isometies' (In this quote, the symbol α^A stands for what we write as ψ.) The question may thus be asked as to how our framework manages to avoid the apparent necessity of restricting attention to a particular vector field X in order to arrive at a meaningful definition of $\mathcal{L}_X\psi$ (or to a particular spacetime connection in the case of $\nabla_X\psi$). More precisely, we shall investigate hereafter, in the present part II, why our formalism of part I is compatible, in general, with tensor calculus in contrast with the other frameworks for spinorial Lie and covariant derivatives available in the literature.

We shall prove that the reason for which our formalism of part I is compatible with tensor calculus, without any restriction on the spacetime connection (or on the vector field X for the Lie derivative), is as follows. To establish the claim reproduced in the above quote [2], it is necessary that a certain hypothesis be satisfied, namely, that the covariant derivative ∇_X (or the Lie derivative $\mathcal{L}_X$) should commute with the map Δ^{-1} that enables one to define the tensorial equivalent of a spinor. (See section 3 for details.) Our formalism of part I does *not* impose this requirement of commutation and, as a consequence, is compatible with tensor calculus without restriction on the spacetime connection (or the vector field X for the Lie derivative). In other words, if one insists on a concept of derivative which enjoys the property of commuting with the map Δ^{-1}, then one *must* restrict attention to particular

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spacetime connections (or particular vector fields X for the Lie derivative); on the other hand, if one is prepared to abandon the hypothesis of commutation, the necessity for any restriction disappears. Our construction is designed to be consistent with tensor calculus even in the non-commuting case and thus, contains the conventional formalism as a special case.

The same statement may also be formulated in another manner. Consider the space of tensors Θ, the space of spinors Σ and the above-mentioned map Δ relating Θ to Σ. (See section 3 for details.) Let the covariant derivatives on Θ and Σ be denoted by ∇ and $\tilde{\nabla}$, respectively. (In part I, we used the same symbol ∇ for both the tensorial and the spinorial covariant derivative but it will prove clearer to employ henceforth two distinct symbols.) It is then possible to ask the question as to whether the following diagram commutes (for every vector field X):

$$\begin{array}{ccc} \Theta & \xrightarrow{\Delta} & \Sigma \\ \nabla_X\downarrow & & \downarrow\tilde{\nabla}_X \\ \Theta & \xrightarrow{\Delta} & \Sigma . \end{array} \tag{1.1}$$

The commutation of (1.1) would be equivalent to the fact that, for every tensor τ and vector X belonging to Θ, one would have

$$\nabla_X \tau = \Delta^{-1}\{\tilde{\nabla}_X[\Delta(\tau)]\}. \tag{1.2}$$

We shall establish that (1.1) does *not* commute and that (1.2) does *not* hold, unless the connection is conformal. The various formalisms for the covariant derivative of spinor fields available in the literature assume (sometimes implicitly) commutation of (1.1) and, therefore, must be restricted to conformal connections. Our formalism abandons the commutation hypothesis and thus does not require restriction of the connection. It is important to emphasize that the fact that, in general, (1.1) is not commuting does *not* imply that our spinorial framework of part I is inconsistent with tensor calculus†, as will be seen in sections 4 and 5.

These considerations will be presented in five steps: in section 2, we shall make a special choice of basis in spinor space which simply amounts to selecting, for convenience, the standard Infeld–van der Waerden two-component spinor formalism [3–6] to perform the calculations that will follow. We shall show how to express our framework of part I in the two-component language. Then, in section 3, we shall define the map Δ^{-1} that relates spinors to tensors and its covariant derivative $\tilde{\nabla}_X\Delta^{-1}$ will be calculated in section 4. All these results will be combined in sections 4 and 5 to investigate the question of the compatibility with tensor calculus and of the non-commutation of $\tilde{\nabla}_X$ and Δ^{-1}, or equivalently of the non-commutation of diagram (1.1). Finally, in section 6, a geometrical interpretation of this non-commutation will be provided in terms of the concept [7] of 'flag pole' of a spinor.

Following the method adopted in part I, we shall exclusively develop here, in detail, the formalism of the covariant derivative $\tilde{\nabla}_X$. There is no difficulty in re-interpreting the construction as a method for defining a Lie derivative $\mathcal{L}_X$ and explicit 'translation rules' for performing this re-interpretation are found in appendix 1 of part I.

† The authors thank the referee for indicating the desirability of emphasizing this point.

2. Choice of basis

In part I, we introduced the bundle $PS^+(\mathcal{M})$ of spin frames above a manifold $\mathcal{M}$ as the double covering of $PO^+(\mathcal{M})$, the bundle of (positively oriented) orthonormal frames above $\mathcal{M}$. Their structure groups are, respectively, the spin group $SP(4)$, which is a subset of the real Clifford algebra $C_{3,1}(\mathbb{R})$, and the special orthogonal group $SO(3,1)$, with Lie algebras denoted by $sp(4)$ and $so(3,1)$, respectively. (See part I for details.) An assignment of a family of spin frames $\tilde{e}_{(M)}$ over $\mathcal{M}$ is then a section σ of $PS^+(\mathcal{M})$ and a spinor field ψ, a linear combination of the basic vectors of σ

$$\psi \equiv \psi^M \tilde{e}_{(M)}. \tag{2.1}$$

Furthermore, the covariant derivative $\tilde{\nabla}_X\psi$ of ψ was defined as [1]

$$\begin{aligned}
&\tilde{\nabla}_X\psi \equiv \left[X(\psi^M) - \tilde{A}^M{}_N(X)\psi^N\right]\tilde{e}_{(M)} \\
&-2\tilde{A}^M{}_N(X) \equiv {}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N + \frac{k}{4}\eta^{\mu\nu}\,{}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\delta^M{}_N \\
&4(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N \equiv \gamma^{\hat{\mu}M}{}_P\gamma^{\hat{\nu}P}{}_N - \gamma^{\hat{\nu}M}{}_P\gamma^{\hat{\mu}P}{}_N \\
&\gamma_{e^{(\hat{\mu})}} \equiv \gamma^{\hat{\mu}M}{}_N\tilde{e}_{(M)} \otimes \tilde{e}^{(N)}
\end{aligned} \tag{2.2}$$

where ${}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}$ and ${}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}$ denote, respectively, the symmetric and antisymmetric parts of the spacetime connection $\mathcal{A}_{\hat{\mu}\hat{\nu}}$ in an orthonormal frame $e_{(\hat{\alpha})}$, k is a free parameter (to be determined later) and γ is a representation of the Clifford algebra in spinor space so that γ_s is an operator acting on a spinor for every s in $C_{3,1}(\mathbb{R})$. Explicit expressions for $\mathcal{A}_{\hat{\mu}\hat{\nu}}$, ${}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}$ and ${}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}$ in terms of the connection components $\Gamma_{\hat{\mu}\hat{\nu}\hat{\alpha}}$, the non-holonomicity $C_{\hat{\alpha}\hat{\mu}\hat{\nu}}$, the contorsion $Q_{\hat{\alpha}\hat{\mu}\hat{\nu}}$ and the non-metricity $H_{\hat{\alpha}\hat{\mu}\hat{\nu}}$ of spacetime are found in part I, the notation of which we follow here, apart from some minor changes introduced for future convenience.

As mentioned in the introduction, all our considerations will be presented in the Infeld–van der Waerden [3–6] two-component spinor formalism. In order to relate it to the formalism of part I, more information is required about representations of the Clifford algebra $C_{3,1}(\mathbb{R})$. In particular, we need to know the relationship between the matrices $\gamma^{\hat{\mu}M}{}_N$ of the representation γ used in (2.2) and the Infeld–van der Waerden symbols $\sigma^{\hat{\mu}\dot{a}}{}_b$ of the two-component formalism [7]. We must also clarify the link between the spinors defined in part I, such as ψ of (2.2) above, and the two-component spinors. In the present section, we shall exclusively *state* the translation of (2.1) and (2.2) in two-component language, details about the *construction* of the two-component formalism being available in appendix 1.

The two-component formalism considers, essentially, that spinor space S decomposes as a direct sum of an even part S^+ and an odd part S^- as $S = S^+ \oplus S^-$ with the bases $\tilde{e}_{(M)}$ and $\tilde{e}^{(N)}$ of S and its dual $S^* = (S^+)^* \oplus (S^-)^*$ respectively decomposed as

$$\tilde{e}_{(M)} \equiv \begin{pmatrix}\tilde{e}_{(m)} & \tilde{e}_{(\dot{m})}\end{pmatrix} \qquad \tilde{e}^{(N)} \equiv \begin{pmatrix}\tilde{e}^{(n)} \\ \tilde{e}^{(\dot{n})}\end{pmatrix} \qquad \tilde{e}^{(M)}[\tilde{e}_{(N)}] = \delta^M{}_N. \tag{2.3}$$

A spinor ψ may thus be written as

$$\psi = u^a\tilde{e}_{(a)} + v^{\dot{a}}\tilde{e}_{(\dot{a})} \tag{2.4}$$

where the components u^a and $v^{\dot{a}}$ transform respectively, under a change of basis, by an $SL(2,\mathbb{C})$ matrix and its complex conjugate.

Spinor space S also possesses a metric tensor $\mathcal{G}$ which reads

$$\mathcal{G} \equiv \mathcal{H} + \overline{\mathcal{H}} \qquad \mathcal{H} \equiv \epsilon_{ab}\tilde{e}^{(a)} \otimes \tilde{e}^{(b)} \tag{2.5}$$

where ϵ_{ab} is the antisymmetric matrix

$$\epsilon_{ab} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.6}$$

and the bar denotes complex conjugation. Moreover, the Dirac matrices $\gamma^{\hat{\mu}M}{}_N$ appearing in (2.2) are given in terms of the anti-Hermitian matrices $\sigma^{\hat{\mu}\dot{m}}{}_m$, the Infeld-van der Waerden symbols, by

$$\gamma^{\hat{\mu}M}{}_N = \sqrt{2}\begin{pmatrix} 0 & \overline{\sigma^{\hat{\mu}\dot{m}}{}_n} \\ \sigma^{\hat{\mu}\dot{m}}{}_n & 0 \end{pmatrix}. \tag{2.7}$$

More details about the properties of the symbols $\sigma^{\hat{\mu}\dot{m}}{}_m$ are found in appendices 1 and 2.

We have now at our disposal enough information to establish the link between the framework of part I and the two-component formalism. For instance, by virtue of (2.2), (2.3) and (2.7), the covariant derivative of a basis $\begin{pmatrix} \tilde{e}^{(a)} \\ \tilde{e}^{(\dot{a})} \end{pmatrix}$ of S^*, the dual of spinor space S, reads

$$\begin{aligned} -2\tilde{\nabla}_X\tilde{e}^{(a)} &= {}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)(\sigma^{\hat{\mu}\hat{\nu}})^a{}_b\tilde{e}^{(b)} + \frac{k}{4}\eta^{\mu\nu}\,{}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\tilde{e}^{(a)} \\ -2\tilde{\nabla}_X\tilde{e}^{(\dot{a})} &= {}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)(\sigma^{\hat{\mu}\hat{\nu}})^{\dot{a}}{}_{\dot{b}}\tilde{e}^{(\dot{b})} + \frac{k}{4}\eta^{\mu\nu}\,{}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\tilde{e}^{(\dot{a})} \\ 2(\sigma^{\hat{\mu}\hat{\nu}})^M{}_N &= \begin{pmatrix} \overline{\sigma^{\hat{\mu}\dot{m}}{}_p}\sigma^{\hat{\nu}\dot{p}}{}_n - \overline{\sigma^{\hat{\nu}\dot{m}}{}_p}\sigma^{\hat{\mu}\dot{p}}{}_n & 0 \\ 0 & \sigma^{\hat{\mu}\dot{m}}{}_p\overline{\sigma^{\hat{\nu}\dot{p}}{}_n} - \sigma^{\hat{\nu}\dot{m}}{}_p\overline{\sigma^{\hat{\mu}\dot{p}}{}_n} \end{pmatrix}. \end{aligned} \tag{2.8}$$

These derivatives will play a fundamental role in the forthcoming calculations.

In order to be able to determine how the covariant derivative (2.2), (2.8) circumvents the apparent necessity [2], mentioned in the introduction, of restricting the connection $\mathcal{A}_{\hat{\mu}\hat{\nu}}$ to some special case in order to arrive at a definition of $\tilde{\nabla}_X$ which is compatible with tensor calculus, it is necessary to establish results about the covariant derivative of the map Δ^{-1} that relates spinor space to the space of tensors. Therefore, the following section will be devoted to introducing the concept of Δ^{-1}, the relevant covariant derivative being obtained in section 4.

3. Tensor map Δ^{-1}

From the Infeld–van der Waerden symbols $\sigma^{\hat{\mu}\dot{a}}{}_b$ of (2.7), it is possible to construct a map Δ which relates the cotangent space $T^*\mathcal{M}$ of $\mathcal{M}$ to the space $(S^-)^* \otimes (S^+)^*$ as

$$\Delta(A_{\hat{\mu}}e^{(\hat{\mu})}) \equiv A_{\hat{\mu}}\sigma^{\hat{\mu}}{}_{\dot{a}a}\tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)} \qquad \sigma^{\hat{\mu}}{}_{\dot{a}b} \equiv \sigma^{\hat{\mu}\dot{c}}{}_b\epsilon_{\dot{c}\dot{a}} \tag{3.1}$$

which is equivalently expressed, given the canonical isomorphism between $T^{**}\mathcal{M}$ and $T\mathcal{M}$, as

$$\Delta = \sigma^{\hat{\mu}}{}_{\dot{a}a}e_{(\hat{\mu})} \otimes \tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)}. \tag{3.2}$$

(Spinor indices are raised and lowered using the spinor metric introduced in (2.5) and (2.6).)

Obviously, there is no obstruction to extending Δ as a map from $(T^*\mathcal{M})^{\otimes n}$ to $[(S^-)^* \otimes (S^+)^*]^{\otimes n}$ for any n. If $n = 2$, which is all that will be used explicitly in the forthcoming calculations, this extension reads

$$\Delta = \sigma^{\hat{\mu}}{}_{\dot{a}a}\sigma^{\hat{\nu}}{}_{\dot{b}b}[e_{(\hat{\mu})} \otimes \tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)}] \otimes [e_{(\hat{\nu})} \otimes \tilde{e}^{(\dot{b})} \otimes \tilde{e}^{(b)}]. \tag{3.3}$$

In particular, one may employ Δ to find the image $\Delta(g)$ of the spacetime metric $g \equiv \eta_{\mu\nu}e^{(\hat{\mu})} \otimes e^{(\hat{\nu})}$ as

$$\Delta(g) = \eta_{\mu\nu}\sigma^{\hat{\mu}}{}_{\dot{a}a}\sigma^{\hat{\nu}}{}_{\dot{b}b}\tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)} \otimes \tilde{e}^{(\dot{b})} \otimes \tilde{e}^{(b)}. \tag{3.4}$$

As a result of lemma 2 of appendix 2, this simplifies to

$$\Delta(g) = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}\tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)} \otimes \tilde{e}^{(\dot{b})} \otimes \tilde{e}^{(b)} \tag{3.5}$$

which will be important in section 4.

Finally, if one introduces the matrices $\sigma_{\hat{\mu}}{}^{\dot{a}a}$ satisfying

$$\sigma_{\hat{\mu}}{}^{\dot{a}a}\sigma^{\hat{\nu}}{}_{\dot{a}a} = \delta^{\hat{\nu}}_{\hat{\mu}} \tag{3.6}$$

the map Δ can be inverted, by virtue of (3.3), to the map Δ^{-1} from $[(S^-)^* \otimes (S^+)^*]^{\otimes 2}$ to $(T^*\mathcal{M})^{\otimes 2}$ given by

$$\Delta^{-1} = \sigma_{\hat{\mu}}{}^{\dot{a}a}\sigma_{\hat{\nu}}{}^{\dot{b}b}[e^{(\hat{\mu})} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)}] \otimes [e^{(\hat{\nu})} \otimes \tilde{e}_{(\dot{b})} \otimes \tilde{e}_{(b)}]. \tag{3.7}$$

The same matrices $\sigma_{\hat{\mu}}{}^{\dot{a}a}$ render it possible to extend Δ even further, in an obvious fashion, from $\Theta \equiv (T^*\mathcal{M})^{\otimes n} \otimes (T\mathcal{M})^{\otimes m}$ to $\Sigma \equiv [(S^-)^* \otimes (S^+)^*]^{\otimes n} \otimes [S^- \otimes S^+]^{\otimes m}$ for any n and m. This enables one to relate the space Θ of tensors n times covariant and m times contravariant to the space Σ of spinors. It will be seen in sections 4 and 5 that the non-commutation of Δ^{-1} and $\tilde{\nabla}_X$ is what enables our formalism to be compatible with tensor calculus without restriction on the spacetime connection.

4. Covariant derivative of the map Δ^{-1} and compatibility with tensor calculus

The fundamental idea behind the verification that the covariant derivative (2.2), (2.8) is compatible with tensor calculus without restriction on the spacetime connection is to calculate the covariant derivative $\nabla_X\tau$ of an arbitrary tensor field τ *via* the map Δ between tensors and spinors as

$$\nabla_X\tau = \nabla_X\left\{\Delta^{-1}[\Delta(\tau)]\right\} = (\tilde{\nabla}_X\Delta^{-1})[\Delta(\tau)] + \Delta^{-1}\{\tilde{\nabla}_X[\Delta(\tau)]\}. \tag{4.1}$$

It is important to emphasize that this calculation assumes that the covariant derivative commutes with contractions and satisfies the Leibniz rule.

The right-hand side of (4.1), involving the spinorial operator $\tilde{\nabla}$ defined in (2.8) and the map Δ of section 3, can be evaluated. The result of the left-hand side, however, is known from tensor calculus since the left-hand side involves only the tensorial operator ∇. If the right-hand side agrees with the result from tensor calculus, we shall say that the spinorial covariant derivative $\tilde{\nabla}$ is *compatible with tensor calculus.*

A *different* but related matter is whether, in (4.1), the term $(\tilde{\nabla}_X \Delta^{-1})[\Delta(\tau)]$ *may always be assumed to vanish* (for every τ) without restriction on the spacetime connection. By virtue of (4.1), this would be equivalent to investigating the possible restrictions imposed on the connection by the constraint

$$\begin{aligned} 0 &= \left(\tilde{\nabla}_X \Delta^{-1}\right)[\Delta(\tau)] \\ &= \nabla_X\{\Delta^{-1}[\Delta(\tau)]\} - \Delta^{-1}\{\tilde{\nabla}_X[\Delta(\tau)]\} \\ &\equiv [\tilde{\nabla}_X, \Delta^{-1}]\{\Delta(\tau)\} \end{aligned} \tag{4.2}$$

(with a very slight abuse of notation in the last line). Assuming that (4.2) holds for *every* tensor τ is, of course, equivalent to imposing

$$0 = [\tilde{\nabla}_X, \Delta^{-1}]. \tag{4.3}$$

If it is the case that (4.3) holds, (4.1) simplifies and becomes

$$\nabla_X \tau = \Delta^{-1}\{\tilde{\nabla}_X[\Delta(\tau)]\} \tag{4.4}$$

which is equivalent to the commutation of the following diagram

$$\begin{array}{ccc} \underbrace{(T^*\mathcal{M})^{\otimes n} \otimes (T\mathcal{M})^{\otimes m}} & \xrightarrow{\Delta} & \underbrace{[(S^-)^* \otimes (S^+)^*]^{\otimes n} \otimes [S^- \otimes S^+]^{\otimes m}} \\ \nabla_X \downarrow & & \downarrow \tilde{\nabla}_X \\ \overbrace{(T^*\mathcal{M})^{\otimes n} \otimes (T\mathcal{M})^{\otimes m}} & \xrightarrow{\Delta} & \overbrace{[(S^-)^* \otimes (S^+)^*]^{\otimes n} \otimes [S^- \otimes S^+]^{\otimes m}} \end{array} \tag{4.5}$$

(for every vector field X), as announced in (1.1) and (1.2) of the introduction.

Both the compatibility with tensor calculus (4.1) and the commutation of diagram (4.5), or equivalently the vanishing of the commutator term in (4.2), should be addressed for the most general tensor τ in $(T^*\mathcal{M})^{\otimes n} \otimes (T\mathcal{M})^{\otimes m}$. Such a calculation is very cumbersome to develop in detail and, therefore, we shall only display the results, in the present section 4, when τ is a one-form. (This special tensor τ will then be denoted by A so as to avoid confusion with the general case.) In other words, we assume that $n = 1$ and $m = 0$ in (4.5).

When all the calculations have been preformed, the two terms on the right-hand side of (4.1) with $\tau = A$ read

$$\begin{aligned} (\tilde{\nabla}_X \Delta^{-1})[\Delta(A)] &= \frac{k}{4}\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X) A + \frac{1}{2} X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}} e^{(\hat{\mu})} A^{\hat{\nu}} \\ \Delta^{-1}\{\tilde{\nabla}_X[\Delta(A)]\} &= -\frac{k}{4}\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X) A + [X(A_{\hat{\mu}}) + {}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X) A^{\hat{\nu}}] e^{(\hat{\mu})}. \end{aligned} \tag{4.6}$$

If our formalism is compatible with tensor calculus, the two right-hand sides of (4.6) must add up, according to (4.1), to the correct tensorial expression for $\nabla_X A$. After using the expression for the decomposition of the connection into its symmetric and its antisymmetric part, given in part I, one finds that the terms on the right-hand side of (4.6) add up to

$$[X(A_{\hat{\mu}}) - A_{\hat{\nu}} \Gamma^{\hat{\nu}}{}_{\hat{\mu}\hat{\alpha}} X^{\hat{\alpha}}] e^{(\hat{\mu})} \tag{4.7}$$

which indeed gives the correct tensorial answer for $\nabla_X A$. This settles the question of compatibility with tensor calculus, at least for tensors belonging to $T^*\mathcal{M}$, namely, one-forms.

The problem of the vanishing of the commutator in (4.2) with $\tau = A$, or equivalently the problem of the commutation of (4.5) for $n = 1$ and $m = 0$, is answered by considering (4.6) again: it is possible to establish that the constraint

$$0 = 4(\tilde{\nabla}_X \Delta^{-1})[\Delta(A)] = k\eta^{\mu\nu}\,{}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)A + 2X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}} e^{(\hat{\mu})} A^{\hat{\nu}} \tag{4.8}$$

is satisfied for an arbitrary A iff the spacetime connection is conformal. (This proof requires using again the symmetric and antisymmetric decomposition of the connection found in part I.) In other words, the diagram (4.5) commutes iff the spacetime connection is conformal.

All these results will be analysed in the conclusion. Let us first discuss their extension to more general tensors.

The calculations (4.6)–(4.8) could be repeated when the tensor τ in (4.1) and (4.2) is a vector V belonging to $T\mathcal{M}$, namely, when $n = 0$ and $m = 1$ in (4.5), instead of being a one-form as in the above treatment. Then, the most general tensor could be built from tensor products of one-forms and vectors. (It is always assumed that the Leibniz rule holds for the tensor product, as emphasized after (4.1).) This is a purely technical routine and need not be pursued here in detail.

On the other hand, performing more explicitly the case where the tensor τ in (4.1) and (4.2) is the metric tensor g turns out to be enlightening for two reasons: first, the expressions corresponding to (4.6) are simpler than the latter; and secondly, the constraint corresponding to (4.8) is also more transparent from the point of view of clarifying the special role played by conformal connections (see below). We are thus, hereafter, going to establish the analogues of (4.6) and (4.8) for the metric tensor g. Given that the metric is a two-tensor, this will also give us the opportunity of showing how the Leibniz rule is to be employed.

In order to proceed in a clear fashion, we shall begin by calculating, in the remainder of this section, the values of $\tilde{\nabla}_X[\Delta(g)]$ and $(\tilde{\nabla}_X \Delta^{-1})[\Delta(g)]$. Then, in section 5, these expressions will be combined together, as in (4.1) with $\tau = g$, to check compatibility with tensor calculus and they will also be used to investigate the commutation of diagram (4.5).

The expression $\tilde{\nabla}_X[\Delta(g)]$ is easily obtained from the definition (3.5) of $\Delta(g)$, the covariant derivatives (2.8) of the bases $\tilde{e}^{(a)}$ and $\tilde{e}^{(\dot{a})}$ and the Leibniz rule as follows:

$$\begin{aligned}
-2\tilde{\nabla}_X[\Delta(g)] &= -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}\tilde{\nabla}_X[\tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)} \otimes \tilde{e}^{(\dot{b})} \otimes \tilde{e}^{(b)}] \\
&= -2\epsilon_{ab}\epsilon_{\dot{a}\dot{b}}[(\tilde{\nabla}_X \tilde{e}^{(\dot{a})}) \otimes \tilde{e}^{(a)} \otimes \tilde{e}^{(\dot{b})} \otimes \tilde{e}^{(b)} + \tilde{e}^{(\dot{a})} \otimes (\tilde{\nabla}_X \tilde{e}^{(a)}) \otimes \tilde{e}^{(\dot{b})} \otimes \tilde{e}^{(b)} \\
&\quad + \tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)} \otimes (\tilde{\nabla}_X \tilde{e}^{(\dot{b})}) \otimes \tilde{e}^{(b)} + \tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)} \otimes \tilde{e}^{(\dot{b})} \otimes (\tilde{\nabla}_X \tilde{e}^{(b)})] \\
&= \{k\eta^{\mu\nu}\,{}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\epsilon_{ab}\epsilon_{\dot{a}\dot{b}} \\
&\quad + {}^{\mathrm{A}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)[\epsilon_{ab}((\sigma^{\hat{\mu}\hat{\nu}})_{\dot{b}\dot{a}} - (\sigma^{\hat{\mu}\hat{\nu}})_{\dot{a}\dot{b}}) + \epsilon_{\dot{a}\dot{b}}((\sigma^{\hat{\mu}\hat{\nu}})_{ba} - (\sigma^{\hat{\mu}\hat{\nu}})_{ab})]\} \\
&\quad \times \tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)} \otimes \tilde{e}^{(\dot{b})} \otimes \tilde{e}^{(b)} \\
&= k\eta^{\mu\nu}\,{}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\Delta(g)
\end{aligned} \tag{4.9}$$

where, in the last step, use has been made of lemma 3 of appendix 2. One should note how much simpler the final answer of (4.9) is than the corresponding expression for $\tilde{\nabla}_X[\Delta(A)]$ obtainable from (4.6) and valid for a one-form A.

A similar derivation yields $\tilde{\nabla}_X \Delta^{-1}$; it is more cumbersome than that leading to (4.9), merely because of the five tensor products appearing in the definition (3.7) of Δ^{-1}. The treatment can be somewhat shortened if one first calculates the covariant derivative of the restricted inverse map denoted by ${}^{\mathrm{R}}\Delta^{-1}$ to avoid confusions with Δ^{-1} and defined as

$$ {}^{\mathrm{R}}\Delta^{-1} \equiv \sigma_{\hat{\mu}}{}^{\dot{a}a} e^{(\hat{\mu})} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)}. \tag{4.10} $$

This intermediate stage is helpful since the complete map Δ^{-1} of (3.7) is the tensor product of two restricted maps of type ${}^{\mathrm{R}}\Delta^{-1}$.

To calculate $\tilde{\nabla}_X {}^{\mathrm{R}}\Delta^{-1}$, we apply again the Leibniz rule, this time to definition (4.10), use the expressions for the covariant derivatives of the basic vectors $e^{(\hat{\mu})}$, $\tilde{e}_{(\dot{a})}$, $\tilde{e}_{(a)}$ and find that

$$ \begin{aligned} 4\tilde{\nabla}_X {}^{\mathrm{R}}\Delta^{-1} = {} & k\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\, {}^{\mathrm{R}}\Delta^{-1} + \{-4\mathcal{A}_{\hat{\alpha}\hat{\mu}}(X)\sigma^{\hat{\alpha}\dot{a}a} \\ & + 2\, {}^{\mathrm{A}}\mathcal{A}_{\hat{\alpha}\hat{\beta}}(X)[\sigma_{\hat{\mu}}{}^{\dot{m}a}(\sigma^{\hat{\alpha}\hat{\beta}})^{\dot{a}}{}_{\dot{m}} + \sigma_{\hat{\mu}}{}^{\dot{a}m}(\sigma^{\hat{\alpha}\hat{\beta}})^{a}{}_{m}]\} e^{(\hat{\mu})} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)} \end{aligned} \tag{4.11} $$

which may be simplified by employing lemma 4 of appendix 2. The result reads

$$ \begin{aligned} 4\tilde{\nabla}_X {}^{\mathrm{R}}\Delta^{-1} &= k\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\, {}^{\mathrm{R}}\Delta^{-1} - 2[\mathcal{A}_{\hat{\mu}\hat{\nu}} + \mathcal{A}_{\hat{\nu}\hat{\mu}}](X)\sigma^{\hat{\nu}\dot{a}a} e^{(\hat{\mu})} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)} \\ &= k\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\, {}^{\mathrm{R}}\Delta^{-1} + 2X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}}\sigma^{\hat{\nu}\dot{a}a} e^{(\hat{\mu})} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)} \end{aligned} \tag{4.12} $$

where the last step uses the relationship, found in appendix 2 of part I, between the symmetric part of the connection and the non-metricity.

The covariant derivative $\tilde{\nabla}_X \Delta^{-1}$ of the complete map Δ^{-1} is now readily obtained, by virtue of the covariant derivative (4.12) of the restricted map ${}^{\mathrm{R}}\Delta^{-1}$, as

$$ \begin{aligned} 4\tilde{\nabla}_X \Delta^{-1} &= 4\tilde{\nabla}_X ({}^{\mathrm{R}}\Delta^{-1} \otimes {}^{\mathrm{R}}\Delta^{-1}) \\ &= [k\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\, {}^{\mathrm{R}}\Delta^{-1} + 2X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}}\sigma^{\hat{\nu}\dot{a}a} e^{(\hat{\mu})} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)}] \otimes {}^{\mathrm{R}}\Delta^{-1} \\ &\quad + {}^{\mathrm{R}}\Delta^{-1} \otimes [k\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\, {}^{\mathrm{R}}\Delta^{-1} + 2X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}}\sigma^{\hat{\nu}\dot{a}a} e^{(\hat{\mu})} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)}] \\ &= 2k\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)\Delta^{-1} + 2X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\beta}\hat{\mu}}\sigma^{\hat{\beta}\dot{a}a}\sigma_{\hat{\nu}}{}^{\dot{b}b} \\ &\quad \times [e^{(\hat{\mu})} \otimes e^{(\hat{\nu})} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)} \otimes \tilde{e}_{(\dot{b})} \otimes \tilde{e}_{(b)} + e^{(\hat{\nu})} \otimes e^{(\hat{\mu})} \otimes \tilde{e}_{(\dot{b})} \otimes \tilde{e}_{(b)} \otimes \tilde{e}_{(\dot{a})} \otimes \tilde{e}_{(a)}]. \end{aligned} \tag{4.13} $$

In particular, when applied to the spinor $\Delta(g)$, the covariant derivative $\tilde{\nabla}_X \Delta^{-1}$ yields, by (3.5), (3.6) and (4.13),

$$ 2(\tilde{\nabla}_X \Delta^{-1})[\Delta(g)] = k\eta^{\mu\nu}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X) g + 2X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}} e^{(\hat{\mu})} \otimes e^{(\hat{\nu})} \tag{4.14} $$

which should be compared with the expression for $(\tilde{\nabla}_X \Delta^{-1})[\Delta(A)]$ of (4.6), valid for a one-form A.

We are now in the position to combine all our results together to investigate the questions of the compatibility with tensor calculus and of the non-commutation of Δ^{-1} and $\tilde{\nabla}_X$, both in the case of the metric, i.e. $\tau = g$ in (4.1) and (4.2). This will be performed in the next section.

5. Compatibility with tensor calculus and the non-commutation of $\tilde{\nabla}_X$ and Δ^{-1} for the metric

Having calculated, in (4.14) and (4.9), expressions for $(\tilde{\nabla}_X\Delta^{-1})[\Delta(g)]$ and $\tilde{\nabla}_X[\Delta(g)]$, respectively, it is now possible to substitute them in (4.1), with $\tau = g$, to find

$$\begin{aligned} 4\nabla_X g &= 4(\tilde{\nabla}_X\Delta^{-1})[\Delta(g)] + 4\Delta^{-1}\{\tilde{\nabla}_X[\Delta(g)]\} \\ &= 4[\tilde{\nabla}_X, \Delta^{-1}][\Delta(g)] + 4\Delta^{-1}\{\tilde{\nabla}_X[\Delta(g)]\} \\ &= [2k\eta^{\beta\gamma}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\beta}\hat{\gamma}}(X)g + 4X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}} e^{(\hat{\mu})} \otimes e^{(\hat{\nu})}] + [-2k\eta^{\beta\gamma}\, {}^{\mathrm{S}}\mathcal{A}_{\hat{\beta}\hat{\gamma}}(X)g] \\ &= 4X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}} e^{(\hat{\mu})} \otimes e^{(\hat{\nu})}. \end{aligned} \tag{5.1}$$

As one can see, the final result is identical to that provided by tensor calculus, namely, $\nabla_X g = X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}} e^{(\hat{\mu})} \otimes e^{(\hat{\nu})}$ given in appendix 2 of part I and this is the case without any restriction on the spacetime connection. (Compatibility with tensor calculus is also much more obvious in the case (5.1) of the metric than in the case (4.6) of a one-form, which is why we presented the details of the former rather than of the latter.) Note that this compatibility with tensor calculus is achieved independently of the value of k. In particular, the covariant derivative with $k = 0$, used by some previous authors such as [8–12], is compatible with tensor calculus.

In order to identify now the most general type of connection which yields a vanishing commutator term $[\tilde{\nabla}_X, \Delta^{-1}][\Delta(g)]$ in (5.1), or a commuting diagram (4.5), we return to (5.1) and solve the constraint $0 = [\tilde{\nabla}_X, \Delta^{-1}][\Delta(g)]$ for the non-metricity H. After substitution of ${}^{\mathrm{S}}\mathcal{A}_{\hat{\mu}\hat{\nu}}(X)$ by its value $-X^{\hat{\alpha}} H_{\hat{\alpha}\hat{\mu}\hat{\nu}}/2$ from (3.12) of part I, there follows

$$0 = X^{\hat{\alpha}}(-k\eta^{\beta\gamma} H_{\hat{\alpha}\hat{\beta}\hat{\gamma}}\eta_{\mu\nu} + 4H_{\hat{\alpha}\hat{\mu}\hat{\nu}})e^{(\hat{\mu})} \otimes e^{(\hat{\nu})}. \tag{5.2}$$

If $k \neq 1$, the constraint is only satisfied (independently of X) by the trivial solution $H_{\hat{\alpha}\hat{\mu}\hat{\nu}} = 0$ which implies that the connection must be metric-compatible. On the other hand, if $k = 1$, the unique solution of (5.2) is

$$H_{\hat{\alpha}\hat{\mu}\hat{\nu}} = V_{\hat{\alpha}}\eta_{\mu\nu} \tag{5.3}$$

for an arbitrary V, which corresponds to a conformal connection.

We have thus, simultaneously clarified the exceptional nature of conformal connections and shown that, among the family of derivatives (2.2) depending on the parameter k, one member, namely, that with $k = 1$, is singled out by the property that it allows the commutator $[\tilde{\nabla}_X, \Delta^{-1}]$ to vanish and diagram (4.5) to commute in the case of a conformal connection. In general, however, i.e. in the case of a non-conformal connection, the commutator is crucially non-vanishing and diagram (4.5) non-commuting. (Indeed, were it not for the presence of the commutator term in (5.1), our framework would be, in general, inconsistent with tensor calculus.) This algebraic fact has a simple geometrical interpretation in terms of the 'flag pole' of a spinor under parallel transport. The following section will be devoted to relating the commutator to properties of the parallel transport.

6. Geometrical interpretation of the non-commutation

In section 3, we introduced a map Δ relating tensor space to spinor space and its inverse Δ^{-1}. (In the restricted case of a covariant tensor of rank 1, belonging to $T^*\mathcal{M}$, and a spinor belonging to the space $(S^-)^* \otimes (S^+)^*$, these maps were denoted by ${}^{\mathrm{R}}\Delta$ and ${}^{\mathrm{R}}\Delta^{-1}$, respectively.) By virtue of (4.10), the tensor associated to the spinor

$$\tilde{A} \equiv A_{\dot{a}a}\tilde{e}^{(\dot{a})} \otimes \tilde{e}^{(a)} \tag{6.1}$$

by Δ^{-1} (or ${}^{\mathrm{R}}\Delta^{-1}$) reads

$$A \equiv \Delta^{-1}(\tilde{A}) = {}^{\mathrm{R}}\Delta^{-1}(\tilde{A}) = e^{(\hat{\mu})}\sigma_{\hat{\mu}}{}^{\dot{a}a}A_{\dot{a}a}. \tag{6.2}$$

In particular, it makes sense to apply Δ^{-1} to the special elements of $(S^-)^* \otimes (S^+)^*$ which are of the form

$$\tilde{A} = \overline{u} \otimes u \qquad u \equiv u_a\tilde{e}^{(a)}. \tag{6.3}$$

In this context, the unique one-form A associated to $\tilde{A} = \overline{u} \otimes u$ by Δ^{-1} is called [7] the 'flag pole' of u. This concept of flag pole illuminates the problem of the non-commutation, in general, of $\tilde{\nabla}_X$ and Δ^{-1} as will now be seen.

Consider a vector field X and a spinor field $u \equiv u_a\tilde{e}^{(a)}$. The covariant derivative of A, the flag pole of u, may then be calculated as

$$\nabla_X A = \nabla_X[\Delta^{-1}(\overline{u} \otimes u)] = (\tilde{\nabla}_X\Delta^{-1})(\overline{u} \otimes u) + \Delta^{-1}[\tilde{\nabla}_X(\overline{u} \otimes u)]. \tag{6.4}$$

Therefore, if u is parallel-transported along the integral curves Γ^X of X, the second term on the right-hand side of (6.4) drops out and the flag pole of u satisfies, by (4.1) with $\tau = \Delta^{-1}(\overline{u} \otimes u)$,

$$\nabla_X A = [\tilde{\nabla}_X, \Delta^{-1}](\overline{u} \otimes u). \tag{6.5}$$

It follows that, unless the commutator vanishes, the flag pole is *not* parallel-transported along Γ^X. In other words, the flag pole of the parallel-transported spinor does *not* agree with the parallel transport of the flag pole of the original spinor unless the commutator vanishes. This means that the operations 'parallel transport' and 'flag-pole taking' commute iff $[\tilde{\nabla}_X, \Delta^{-1}] = 0$, which happens, as proved in section 5, iff the connection is conformal. We thus, have a geometrical interpretation of the difference between the framework introduced in part I for the covariant derivative and other formalisms, on which we shall elaborate in the conclusion.

7. Conclusion

In this work, we investigated further the framework developed [1] in part I for defining the covariant derivative of a spinor field. The particular characteristic of this formalism lies in the fact that its concept of derivative is meaningful without restriction on the spacetime connection.

We then introduced, in section 3, a map Δ^{-1} which relates spinor space to tensor space in such a way that it becomes possible to ask the question as to whether the *spinorial* covariant derivative of part I is consistent with *tensor* calculus. In sections 4 and 5, we checked that the construction was indeed consistent with tensor calculus for an arbitrary connection. This contrasts with statements found in the literature (e.g. in [2]) and according to which the concept of covariant derivative requires the connection to be conformal.

The origin of the difference between our framework and others was also investigated in sections 4 and 5. It was found that the apparent necessity of restricting attention to conformal connections is circumvented by our formalism since the latter does not possess, in general, one of the properties assumed (sometimes implicitly) in other approaches, namely, the commutation of the covariant derivative $\tilde{\nabla}_X$ and the above-mentioned map Δ^{-1} or equivalently the commutation of diagram (4.5).

This non-commutation was interpreted geometrically in section 6 by making use of the notion of 'flag pole' of a spinor [7]. It was shown that, when a spinor is parallel-transported according to our method, its flag pole, in general, is not parallel-transported unless the covariant derivative commutes with the map Δ^{-1}. If one insists on imposing parallel transport of the flag pole during parallel transport of the spinor, then it is necessary and sufficient to restrict attention to conformal connections, which is normally performed in the literature [2]. In other words, our framework is consistent with tensor calculus in all cases, i.e. for an arbitrary connection (in contrast with the other formalisms), and this is achieved at the expense of abandoning, in general, the requirement of commutation of flag-pole taking and parallel transport. The only instance in which these operations may consistently be assumed to commute is when the transport is performed by a conformal connection.

The appearance of a conformal connection as an exceptional case is consistent with the fact that the flag pole of any spinor u in S^+ is null [7]. Consequently, if, during parallel transport of a spinor, the flag pole is also to be parallel-transported, the connection must respect null vectors and thus, be conformal. It is, however, by no means necessary, from the mathematical point of view, to adopt this requirement of parallel transportation of flag poles; we investigated here, and in part I, a definition of the covariant derivative which does *not* impose it and this does *not* lead to a self-contradiction or a contradiction with tensor calculus as shown in sections 4 and 5. This generalized concept of derivative would be necessary, for instance, to express consistently the Dirac equation in a spacetime with a connection more general than a conformal one. Another possible physical application would be the study of symmetries, *via* the Lie derivative, of classical theories involving spinors, as performed, for instance, in [12].

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Appendix 1. The Infeld–van der Waerden two-component formalism

In order to clarify the relationship between our formalism of part I and the standard Infeld–van der Waerden two-component formalism [3–6], we need some information about the Clifford algebra $C_{3,1}(\mathbb{R})$. We shall, therefore, hereafter, exclusively develop what is necessary for our purposes, the reader being referred to the literature (e.g. [5]) for algebraic details about $C_{3,1}(\mathbb{R})$.

The regular representation ρ maps $C_{3,1}(\mathbb{R})$ into its endomorphism algebra as

$$\rho : \qquad s \longrightarrow \rho_s : \qquad \rho_s(x) \equiv s \vee x \qquad \forall s, x \in C_{3,1}(\mathbb{R}) \qquad \text{(A.1)}$$

where $\vee$ is the Clifford product. This representation is reducible and its invariant subspaces are the left ideals of $C_{3,1}(\mathbb{R})$. Therefore, by definition, ρ is *irreducible* on any *minimal* left ideal of $C_{3,1}(\mathbb{R})$. Different choices of a minimal left ideal lead to different, but equivalent, representations and, therefore, no generality is lost in making any particular choice. When a choice has been made, the minimal left ideal selected is called spinor space (denoted by S in part I) and an element of this space is called a spinor.

Consider now the complexification $C_{3,1}^{\mathbb{C}}$ of $C_{3,1}(\mathbb{R})$. The following isomorphisms hold [5]

$$C_{3,1}^{\mathbb{C}} \simeq M_4(\mathbb{C}) \qquad C_{3,1}(\mathbb{R}) \simeq M_4(\mathbb{R}) \tag{A.2}$$

in which the symbol $M_n(\mathbb{F})$ stands for the algebra of square matrices of order n over the field $\mathbb{F}$. Moreover, the even subalgebras $C_{3,1}^{\mathbb{C}+}$ and $C_{3,1}^{+}(\mathbb{R})$ of $C_{3,1}^{\mathbb{C}}$ and $C_{3,1}(\mathbb{R})$ satisfy

$$C_{3,1}^{\mathbb{C}+} \simeq M_2(\mathbb{C}) \oplus M_2(\mathbb{C}) \qquad C_{3,1}^{+}(\mathbb{R}) \simeq M_2(\mathbb{C}). \tag{A.3}$$

Therefore, the matrix structure of $C_{3,1}^{\mathbb{C}}$ and $C_{3,1}^{\mathbb{C}+}$ enables one to choose a matrix basis for $C_{3,1}^{\mathbb{C}}$ in which the elements of $C_{3,1}^{\mathbb{C}+}$ are block diagonal, the off-diagonal components corresponding to odd elements of $C_{3,1}^{\mathbb{C}}$. We shall thus write an element s of $C_{3,1}^{\mathbb{C}}$ as

$$s \equiv \begin{pmatrix} E_1 & O_1 \\ O_2 & E_2 \end{pmatrix} \tag{A.4}$$

where E_i and O_i $(i = 1, 2)$ are complex 2×2 matrices referring, respectively, to the even and odd elements of $C_{3,1}^{\mathbb{C}}$.

Furthermore, it is a simple matter to check that the elements of $C_{3,1}^{\mathbb{C}}$ of the form

$$L \equiv \begin{pmatrix} u & \mathbf{0} \\ \dot{u} & \mathbf{0} \end{pmatrix} \tag{A.5}$$

in which u and $\dot{u}$ denote complex 2-columns and $\mathbf{0}$ is an abbreviation for the 2×3 null matrix, form a minimal left ideal $\mathcal{I}$ of $C_{3,1}^{\mathbb{C}}$. As explained above, no generality is lost by adopting this particular minimal left ideal $\mathcal{I}$ as spinor space S. Obviously, L of (A.5) is isomorphic to the complex 4-column

$$\psi \equiv \begin{pmatrix} u \\ \dot{u} \end{pmatrix} \tag{A.6}$$

and we shall thus, from now on, use this notation instead of the more cumbersome one of (A.5) to denote the components of a spinor ψ.

By virtue of (A.1), (A.4) and (A.6), the regular representation ρ acts on a spinor ψ, an element of $\mathcal{I}$, as

$$\rho_s \psi \equiv \rho_{\begin{pmatrix} E_1 & O_1 \\ O_2 & E_2 \end{pmatrix}} \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} E_1 & O_1 \\ O_2 & E_2 \end{pmatrix} \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} E_1 u + O_1 \dot{u} \\ O_2 u + E_2 \dot{u} \end{pmatrix} \qquad s \in C_{3,1}^{\mathbb{C}} \tag{A.7}$$

where a component notation is understood as in (A.6). In particular, if s is a purely even element of $C_{3,1}^{\mathbb{C}}$, the matrices O_1 and O_2 vanish so that

$$\rho_s \psi = \rho_s \begin{pmatrix} u \\ \dot{u} \end{pmatrix} = \begin{pmatrix} E_1 u \\ E_2 \dot{u} \end{pmatrix} \tag{A.8}$$

which proves that the components u and $\dot{u}$ transform independently of one another. They correspond, respectively, by definition, to the even and odd parts of ψ, each of them forming a so-called 'two-component' or 'Weyl' spinor. (In this terminology, the total 'four-component' spinor ψ is called a 'Dirac' spinor.)

In the subspaces S^+ of the even part and S^- of the odd part, we may denote a basis, respectively, by $\tilde{e}_{(m)}$ and $\tilde{e}_{(\dot{m})}$, m, $\dot{m} = 1, 2$ and so construct a basis $\tilde{e}_{(M)}$ of spinor space $S = S^+ \oplus S^-$ as

$$\tilde{e}_{(M)} \equiv (\,\tilde{e}_{(m)} \quad \tilde{e}_{(\dot{m})}\,) \tag{A.9}$$

with the corresponding dual basis $\tilde{e}^{(N)}$ of $S^* = (S^+)^* \oplus (S^-)^*$ being given by

$$\tilde{e}^{(N)} \equiv \begin{pmatrix} \tilde{e}^{(n)} \\ \tilde{e}^{(\dot{n})} \end{pmatrix} \qquad \tilde{e}^{(M)}[\tilde{e}_{(N)}] = \delta^M{}_N. \tag{A.10}$$

With this notation, the representation (A.8) may be rewritten as

$$\rho_s\psi = \rho_s[\psi^M\tilde{e}_{(M)}] = \rho_s[u^a\tilde{e}_{(a)} + u^{\dot{a}}\tilde{e}_{(\dot{a})}] = E^a_1{}_b u^b\tilde{e}_{(a)} + E^{\dot{a}}_2{}_{\dot{b}}u^{\dot{b}}\tilde{e}_{(\dot{a})} \tag{A.11}$$

or, equivalently, as

$$\rho_s = E^a_1{}_b\tilde{e}_{(a)} \otimes \tilde{e}^{(b)} + E^{\dot{a}}_2{}_{\dot{b}}\tilde{e}_{(\dot{a})} \otimes \tilde{e}^{(\dot{b})} \qquad s \in C^{\mathbb{C}+}_{3,1}. \tag{A.12}$$

It is important to emphasize that we are, in fact, interested in the real Clifford algebra $C_{3,1}(\mathbb{R})$ and its (real) even subalgebra $C^+_{3,1}(\mathbb{R})$; we are only embedding them in a convenient fashion in the complex algebras $C^{\mathbb{C}}_{3,1}$ and $C^{\mathbb{C}+}_{3,1}$. Moreover, by (A.3), the complex even subalgebra $C^{\mathbb{C}+}_{3,1}$ contains twice as many free constants as the real even subalgebra $C^+_{3,1}(\mathbb{R})$. Thus, in order to make half of $C^{\mathbb{C}+}_{3,1}$ redundant, one may impose the condition that the two complex matrices of $C^{\mathbb{C}+}_{3,1}$ be complex conjugates of each other. A similar requirement imposed on the representation (A.12) of $C^{\mathbb{C}+}_{3,1}$ yields then the constraints

$$\overline{E^{\dot{a}}_2{}_{\dot{b}}} = E^a_1{}_b \equiv E^a{}_b \qquad \overline{\tilde{e}_{(\dot{a})}} = \tilde{e}_{(a)} \qquad \overline{\tilde{e}^{(\dot{a})}} = \tilde{e}^{(a)} \tag{A.13}$$

in which the bar denotes complex conjugation.

By the same treatment as that leading from (A.7) to (A.12), it is easy to show that the operator on spinor space associated by ρ to the (real and purely odd) basic vector $e^{(\hat{\mu})}$ is

$$2^{-1/2}\rho_{e^{(\hat{\mu})}} = \sigma^{\hat{\mu}\dot{a}}{}_b\tilde{e}_{(\dot{a})} \otimes \tilde{e}^{(b)} + \overline{\sigma^{\hat{\mu}\dot{a}}{}_b}\tilde{e}_{(a)} \otimes \tilde{e}^{(\dot{b})} \tag{A.14}$$

for certain 2×2 complex matrices $\sigma^{\hat{\mu}\dot{a}}{}_b$. (The factor $2^{-1/2}$ has been inserted for future convenience.) In part I and in (2.2) above, we used the symbol γ to indicate a representation of $C_{3,1}(\mathbb{R})$ in spinor space S and, in particular, $\gamma^{\hat{\mu}} \equiv \gamma_{e^{(\hat{\mu})}}$ was the operator on S associated with the basic vector $e^{(\hat{\mu})}$. It follows from (A.14) that, with our choice of the minimal left ideal $\mathcal{I}$ as the spinor space S, the operator $\gamma^{\hat{\mu}}$ reads

$$2^{-1/2}\gamma^{\hat{\mu}} = \sigma^{\hat{\mu}\dot{a}}{}_b\tilde{e}_{(\dot{a})} \otimes \tilde{e}^{(b)} + \overline{\sigma^{\hat{\mu}\dot{a}}{}_b}\tilde{e}_{(a)} \otimes \tilde{e}^{(\dot{b})} \tag{A.15}$$

or, in a matrix form, based on (A.9) and (A.10), as

$$\gamma^{\hat{\mu}M}{}_N = \sqrt{2}\begin{pmatrix} 0 & \overline{\sigma^{\hat{\mu}\dot{m}}{}_n} \\ \sigma^{\hat{\mu}\dot{m}}{}_n & 0 \end{pmatrix}. \tag{A.16}$$

The decomposition (A.16) of the Dirac matrices $\gamma^{\hat{\mu}M}{}_N$ enables one now to recognize the complex matrices $\sigma^{\hat{\mu}\dot{m}}{}_n$ as the Infeld–van der Waerden symbols of the two-component formalism. Indeed, by definition of a Clifford algebra, the Dirac matrices (A.16) must satisfy [1]

$$\gamma^{\hat{\mu}M}{}_P\gamma^{\hat{\nu}P}{}_N + \gamma^{\hat{\nu}M}{}_P\gamma^{\hat{\mu}P}{}_N = 2\eta^{\mu\nu}\delta^M{}_N. \tag{A.17}$$

When this equation is translated into a constraint on $\sigma^{\hat{\mu}\dot{m}}{}_n$, namely

$$\sigma_{\hat{\mu}\dot{c}}{}^a\sigma_{\hat{\nu}b}{}^{\dot{c}} + \sigma_{\hat{\nu}\dot{c}}{}^a\sigma_{\hat{\mu}b}{}^{\dot{c}} = -\delta^a{}_b\eta_{\mu\nu} \tag{A.18}$$

it is identical to that imposed on $\sigma^{\hat{\mu}\dot{m}}{}_n$ by the two-component formalism [7] in such a way that the identification of $\sigma^{\hat{\mu}\dot{m}}{}_n$ with the Infeld–van der Waerden symbols is legitimate. The detailed calculation of the constraint, which is purely technical, can be found in lemma 1 of appendix 2.

In the construction so far, we have considered the real even subalgebra $C^+_{3,1}(\mathbb{R})$. In particular, the matrix $E^a{}_b$ of (A.12) and (A.13) carries a representation of $C^+_{3,1}(\mathbb{R}) \equiv M_2(\mathbb{C})$. However, the spin group of $C^+_{3,1}(\mathbb{R})$ is isomorphic [5] to $SL(2,\mathbb{C})$ and it is, therefore, permitted to take $E^a{}_b$ as an $SL(2,\mathbb{C})$ matrix in (A.12) and (A.13). This restriction allows one to define a metric tensor $\mathcal{G}$ on spinor space $S = S^+ \oplus S^-$ as

$$\mathcal{G} \equiv \mathcal{H} + \overline{\mathcal{H}} \qquad \mathcal{H} \equiv \epsilon_{ab}\tilde{e}^{(a)} \otimes \tilde{e}^{(b)} \tag{A.19}$$

in which ϵ_{ab} is the antisymmetric matrix

$$\epsilon_{ab} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{A.20}$$

Such a metric is admissible since the tensor components ϵ_{ab}, albeit not invariant under $M_2(\mathbb{C})$, are invariant under $SL(2,\mathbb{C})$.

Appendix 2. Properties of Infeld–van der Waerden symbols

In the main text, four lemmas concerning the Infeld–van der Waerden symbols play an important role. This appendix is devoted to giving the main steps of the proofs of these lemmas, details being left to the reader.

Lemma 1.

$$\sigma_{\hat{\mu}\dot{c}}{}^a\sigma_{\hat{\nu}b}{}^{\dot{c}} + \sigma_{\hat{\nu}\dot{c}}{}^a\sigma_{\hat{\mu}b}{}^{\dot{c}} = -\delta^a{}_b\eta_{\mu\nu}. \tag{B.1}$$

Proof. From the Clifford-algebra relation (A.17), applied to the realization (A.16) of the Dirac matrices, there follows, after raising an index by (A.19) and (A.20),

$$\begin{aligned}\eta^{\mu\nu}\delta^a{}_b &= \overline{\sigma^{\hat{\mu}a\dot{}}{}_c}\sigma^{\hat{\nu}\dot{c}}{}_b + \overline{\sigma^{\hat{\nu}\dot{a}}{}_c}\sigma^{\hat{\mu}\dot{c}}{}_b \\ &= \overline{\sigma^{\hat{\mu}\dot{a}d}}\epsilon_{\dot{d}\dot{c}}\sigma^{\hat{\nu}\dot{c}}{}_b + \overline{\sigma^{\hat{\nu}\dot{a}d}}\epsilon_{\dot{d}\dot{c}}\sigma^{\hat{\mu}\dot{c}}{}_b.\end{aligned} \tag{B.2}$$

We now make the hypothesis that the Infeld–van der Waerden symbols are anti-Hermitian, namely

$$\overline{\sigma^{\hat{\mu}\dot{a}b}} = -\sigma^{\hat{\mu}\dot{b}a}. \tag{B.3}$$

This comes from the fact [5] that $C_{3,1}(\mathbb{R})$ admits an involutory anti-automorphism ξ defined as

$$\xi[e^{(\hat{\mu}_1)} \otimes e^{(\hat{\mu}_2)} \otimes \ldots \otimes e^{(\hat{\mu}_n)}] = e^{(\hat{\mu}_n)} \otimes e^{(\hat{\mu}_{n-1})} \otimes \ldots \otimes e^{(\hat{\mu}_1)}. \tag{B.4}$$

The anti-Hermiticity of $\sigma^{\hat{\mu}\dot{a}b}$ is then a consequence [5] of (A.14) and the invariance of $e^{(\hat{\mu})}$ under ξ. By virtue of (B.3), it is possible to transform (B.2) into

$$\eta^{\mu\nu}\delta^{a}{}_{b} = -\sigma^{\hat{\mu}\dot{d}a}\epsilon_{\dot{d}\dot{c}}\sigma^{\hat{\nu}\dot{c}}{}_{b} - \sigma^{\hat{\nu}\dot{d}a}\epsilon_{\dot{d}\dot{c}}\sigma^{\hat{\mu}\dot{c}}{}_{b} \tag{B.5}$$

which implies (B.1). □

This lemma is what enables one to interpret the quantities $\sigma^{\hat{\mu}\dot{a}}{}_{b}$ of (A.16) as the Infeld–van der Waerden symbols since (B.1) is precisely the constraint imposed on these symbols by the two-component formalism [7].

Lemma 2.

$$\eta_{\mu\nu}\sigma^{\hat{\mu}}{}_{\dot{a}a}\sigma^{\hat{\nu}}{}_{\dot{b}b} = \epsilon_{ab}\epsilon_{\dot{a}\dot{b}}. \tag{B.6}$$

Proof. The proof is based on the contraction of (B.1) with $\eta^{\mu\nu}\epsilon_{ac}$ and also on the antisymmetry of the metric ϵ_{ab}. Details can be found in [7] since (B.1) is the standard equation on $\sigma^{\hat{\mu}\dot{a}}{}_{a}$ of the two-component formalism. □

Lemma 3.

$$(\sigma^{\hat{\mu}\hat{\nu}})_{\dot{b}\dot{a}} - (\sigma^{\hat{\mu}\hat{\nu}})_{\dot{a}\dot{b}} = (\sigma^{\hat{\mu}\hat{\nu}})_{ba} - (\sigma^{\hat{\mu}\hat{\nu}})_{ab} = 0. \tag{B.7}$$

Proof. As a consequence of the anti-Hermiticity of the Infeld–van der Waerden symbols and the expressions (2.8) for $\sigma^{\hat{\mu}\hat{\nu}}$, one obtains

$$2(\sigma^{\hat{\mu}\hat{\nu}})_{ab} - 2(\sigma^{\hat{\mu}\hat{\nu}})_{ba} = -\sigma^{\hat{\mu}}{}_{\dot{c}a}\sigma^{\hat{\nu}\dot{c}}{}_{b} + \sigma^{\hat{\nu}}{}_{\dot{c}a}\sigma^{\hat{\mu}\dot{c}}{}_{b} + \sigma^{\hat{\mu}}{}_{\dot{c}b}\sigma^{\hat{\nu}\dot{c}}{}_{a} - \sigma^{\hat{\nu}}{}_{\dot{c}b}\sigma^{\hat{\mu}\dot{c}}{}_{a} \tag{B.8}$$

which is manifestly antisymmetrical both in $\hat{\mu}$, $\hat{\nu}$ and in a, b. Moreover, for $\hat{\mu}$ and $\hat{\nu}$ fixed, the right-hand side of (B.8) is an antisymmetrical 2×2 matrix and, therefore, is proportional to ϵ_{ab}. This enables one to write

$$-\sigma^{\hat{\mu}}{}_{\dot{c}a}\sigma^{\hat{\nu}\dot{c}}{}_{b} + \sigma^{\hat{\nu}}{}_{\dot{c}a}\sigma^{\hat{\mu}\dot{c}}{}_{b} + \sigma^{\hat{\mu}}{}_{\dot{c}b}\sigma^{\hat{\nu}\dot{c}}{}_{a} - \sigma^{\hat{\nu}}{}_{\dot{c}b}\sigma^{\hat{\mu}\dot{c}}{}_{a} \equiv F^{\hat{\mu}\hat{\nu}}\epsilon_{ab} \tag{B.9}$$

for a certain antisymmetrical tensor F. After contraction of (B.9) with ϵ^{ab} and the use of (3.6), one evaluates F as

$$2F^{\hat{\mu}\hat{\nu}} = -\eta^{\mu\nu} + \eta^{\nu\mu} - \eta^{\mu\nu} + \eta^{\nu\mu} = 0 \tag{B.10}$$

which establishes the lemma. □

Lemma 4.

$$\sigma_{\hat{\mu}}{}^{\dot{m}a}(\sigma^{\hat{\alpha}\hat{\beta}})^{\dot{a}}{}_{\dot{m}} + \sigma_{\hat{\mu}}{}^{\dot{a}m}(\sigma^{\hat{\alpha}\hat{\beta}})^{a}{}_{m} = \delta^{\hat{\beta}}_{\hat{\mu}}\sigma^{\hat{\alpha}\dot{a}a} - \delta^{\hat{\alpha}}_{\hat{\mu}}\sigma^{\hat{\beta}\dot{a}a}. \tag{B.11}$$

Proof. It is a simple matter to manipulate the Clifford-algebra relation (A.17) to express the product of three Dirac matrices $\gamma^{\hat{\mu}}$ in the well known [13] form

$$2\eta^{\beta\mu}\gamma^{\hat{\alpha}} - 2\eta^{\alpha\mu}\gamma^{\hat{\beta}} = \gamma^{\hat{\alpha}}\gamma^{\hat{\beta}}\gamma^{\hat{\mu}} - \gamma^{\hat{\mu}}\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}} = [\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}}, \gamma^{\hat{\mu}}]. \tag{B.12}$$

Equivalently, this may be written

$$4\eta^{\beta\mu}\gamma^{\hat{\alpha}} - 4\eta^{\alpha\mu}\gamma^{\hat{\beta}} = [(\gamma^{\hat{\alpha}}\gamma^{\hat{\beta}} - \gamma^{\hat{\beta}}\gamma^{\hat{\alpha}}), \gamma^{\hat{\mu}}]. \tag{B.13}$$

After substitution in (B.13) of the realization (A.16) of the Dirac matrices in terms of the Infeld–van der Waerden symbols, a lengthy development exploiting the anti-Hermiticity of $\sigma^{\hat{\mu}\dot{a}a}$ and the antisymmetry of ϵ_{ab} yields

$$2\delta^{\hat{\beta}}_{\hat{\mu}}\sigma^{\hat{\alpha}\dot{a}a} - 2\delta^{\hat{\alpha}}_{\hat{\mu}}\sigma^{\hat{\beta}\dot{a}a} = \sigma_{\hat{\mu}}{}^{\dot{m}a}(-\sigma^{\hat{\alpha}\dot{a}}{}_{n}\sigma^{\hat{\beta}n}{}_{\dot{m}} + \sigma^{\hat{\beta}\dot{a}}{}_{n}\sigma^{\hat{\alpha}n}{}_{\dot{m}}) + \sigma_{\hat{\mu}}{}^{\dot{a}m}(-\sigma^{\hat{\alpha}a}{}_{\dot{n}}\sigma^{\hat{\beta}\dot{n}}{}_{m} + \sigma^{\hat{\beta}a}{}_{\dot{n}}\sigma^{\hat{\alpha}\dot{n}}{}_{m}). \tag{B.14}$$

On the other hand, one may expand the left-hand side of (B.11) in terms of the Infeld–van der Waerden symbols by employing (2.8) for $\sigma^{\hat{\alpha}\hat{\beta}}$ and simplify the result by exploiting again the anti-Hermiticity of $\sigma^{\hat{\mu}\dot{a}}{}_{a}$ and the antisymmetry of ϵ_{ab}. The expression thus obtained is then recognized as the half of the right-hand side of (B.14) and the result follows. □

References

[1] Hurley D and Vandyck M 1994 *J. Phys. A: Math. Gen.* **27** 1

[2] Huggett S and Tod K 1985 *An Introduction to Twistor Theory* (Cambridge: Cambridge University Press) ch 5

[3] van der Waerden B 1929 *Nachr. Akad. Wiss. Götting. Math.-Physik* K **1** 100

[4] Infeld L and van der Waerden B 1933 *Sitz. Ber. Preuss. Akad. Wiss. Physik.-Math.* K **1** 380

[5] Benn I and Tucker R 1988 *An Introduction to Spinors and Geometry* (Bristol: Hilger) ch 2.8, 2.4, 2.5

[6] Ward R and Wells R 1990 *Twistor Geometry and Field Theory* (Cambridge: Cambridge University Press) ch 2.3

[7] Penrose R and Rindler 1987 *Spinors and Spacetime* vol 1 (Cambridge: Cambridge University Press) preface, ch 3.1, 3.2

[8] Lichnerowicz A 1964 *Bull. Soc. Math. France* **92** 11

[9] Kosmann Y 1972 *C. R. Acad. Sci. Paris* **91** 317

[10] Crumeyrolle A 1971 *C. R. Acad. Sci. Paris* **273** 853

[11] Bilyalov R 1992 *Teor. Mat. Fiz.* **90** 369

[12] Vandyck M 1993 *Gen. Relat. Grav.* **25** 15

[13] Aitchison I and Hey A 1989 *Gauge Theories in Particle Physics* (Bristol: Hilger) appendix D